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An extension of Eliezer's theorem on the Abraham–Lorentz–Dirac equation

A Carati

Università di Milano, Dipartimento di Matematica, Via Saldini 50, 20133 Milano, Italy

E-mail: carati@mat.unimi.it

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Abstract

In 1943 Eliezer showed that, according to the Abraham–Lorentz–Dirac equation, a point charge cannot fall on a centre of attractive Coulombian forces, if one considers only motions constrained on a line. In other words, the Abraham–Lorentz–Dirac equation on a line does not admit solutions $x(t)$ such that $x \to 0$ for $t \to t_c$, with either a finite or infinite t_c. In this paper it is shown that this remain true for the full three-dimensional problem.

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1. Introduction

It is known that the motion of a charged point particle is well described, according to classical electromagnetism, by the so-called Abraham–Lorentz–Dirac equation [1, 2]. In the non-relativistic approximation, this takes the form

$$\varepsilon\, \dddot{\boldsymbol{x}} = m\ddot{\boldsymbol{x}} - \boldsymbol{F}(\boldsymbol{x}) \tag{1}$$

where $\boldsymbol{x} \in \mathbb{R}^3$, $\boldsymbol{F}$ is an external mechanical force field, m is the point particle's mass, and the constant ε depends on the charge e of the particle and on the speed of light c through $\varepsilon = \frac{2e^2}{3c^3}$. The term on the left-hand side involving the third derivative is due to the self-interaction between the charge and the electromagnetic field and vanishes for an uncharged particle, for which the more familiar Newton law of motion is recovered.

The Abraham–Lorentz–Dirac equation, being of third order, is non-conservative, and this fact precludes the possibility of providing by elementary mathematical techniques an answer to a problem of great physical interest, such as the stability of the atom. Indeed, if one takes $\boldsymbol{F} = -Ze^2\boldsymbol{x}/|\boldsymbol{x}|^3$, describing the attractive force on an electron due to a nucleus of atomic number Z fixed at the origin, the mechanical energy is not a constant of motion and so it is not known *a priori* whether the particle's motion will be bounded away from the origin or on the contrary will fall on it by spiralling inwards. However, at the beginning of the century the common opinion, mainly based on heuristic considerations, was that a charged particle would fall on the centre of force in a rather short time (10^{-8} s, for example see [3]), and such a lack of accounting for the observed stability of the atoms led to the development of quantum mechanics.

Nevertheless, mathematical theorems on equation (1) were lacking until 1943, when Eliezer [4] (see also [5]) proved a rather astonishing result. Indeed, at variance with what had always been presumed, he proved that, for a motion on a line with an attractive force $-Ze^2/x^2$, there exists no solution such that $x \to 0$ for $t \to t_c$, with a finite or an infinite t_c, namely no solution falls on the centre of force in a finite or an infinite time.

However, from the physical point of view this result is too weak, because it refers to the unrealistic case of motions on a line, namely to a set of initial data of zero measure, having parallel velocity and acceleration, and zero initial angular momentum. Our aim is to extend the theorem to the fully three-dimensional motions, showing that, without any restriction on the initial data, there do not exist motions falling on the centre of force in a finite or an infinite time.

The paper is arranged as follows: in section 2 the main theorem is stated and proved, using two lemmas which are proved in section 3; in section 4 some further comments are added. The proofs of two further lemmas having some general character and used in section 3 are deferred to an appendix.

2. The main theorem

For a mathematical discussion of the solutions of (1), it is convenient to rewrite it in a simpler, dimensionless form; we limit ourselves to the case of an external Coulomb force, mainly the case of the equation $\varepsilon \dddot{\boldsymbol{x}} = m\ddot{\boldsymbol{x}} + \frac{Ze^2\boldsymbol{x}}{|\boldsymbol{x}|^3}$. In terms of $\boldsymbol{x}' = r_0\boldsymbol{x}$, with $r_0 = (4Z/9)^{1/3}e^2/mc^2$ and $t' = m/\varepsilon\, t$ this becomes

$$\dddot{\boldsymbol{x}} = \ddot{\boldsymbol{x}} + \frac{\boldsymbol{x}}{\rho^3}$$

where $\rho = |\boldsymbol{x}|$ and primes are omitted. This is the equation discussed in the rest of the paper. The notation $\boldsymbol{v} = \dot{\boldsymbol{x}}$, $\boldsymbol{a} = \ddot{\boldsymbol{x}}$, and $v = |\boldsymbol{v}|$, $a = |\boldsymbol{a}|$ will often be used. The relevant theorem is the following one.

Theorem. *Consider the differential equation*

$$\dddot{\boldsymbol{x}} = \ddot{\boldsymbol{x}} + \frac{\boldsymbol{x}}{\rho^3} \qquad \textit{with} \quad \rho = |\boldsymbol{x}| \qquad \boldsymbol{x} \in \mathbb{R}^3. \tag{2}$$

Then for any choice of initial data $\boldsymbol{x}_0 \in \mathbb{R}^3/\{0\}$, $\dot{\boldsymbol{x}}_0 \in \mathbb{R}^3$ and $\ddot{\boldsymbol{x}}_0 \in \mathbb{R}^3$ at time t_0, there exists no $t_c \in \mathbb{R} \cup \{+\infty\}$, $t_c > t_0$, such that the corresponding solution $\boldsymbol{x} = \boldsymbol{x}(t)$ verifies

$$\lim_{t \to t_c} \rho(t) = 0. \tag{3}$$

Proof. We will prove the theorem by absurdity, showing that the property $\rho \to 0$ implies $\frac{d}{dt}\rho \to +\infty$, which is impossible for a positive function. More precisely, we will show that there is a contradiction in supposing that there exists a solution of (2) and times t_1, t_c such that

$$\rho > 0 \qquad \text{for} \quad t \in (t_1, t_c) \qquad \text{and} \qquad \lim_{t \to t_c} \rho = 0. \tag{4}$$

This entails that the solution is analytic for $t \in (t_1, t_c)$, which will be used below.

Consider first the case $t_c < +\infty$. Then the following lemma, to be proven in section 3, holds.

Lemma 1. *Let $\boldsymbol{x}(t)$ be a solution of (2) such that $\rho \to 0$ for $t \to t_c$, with $t_c < +\infty$. If ρ^{-1} is integrable on $(t_0, t_c]$, then one has*

(i) $\limsup_{t\to t_c} |\dot\rho| = +\infty$

(ii) $\lim_{t\to t_c} v^2 = +\infty$.

On the other hand, by trivial manipulations one sees that for solutions of (2) one has

$$\rho\ddot\rho = v^2 + 2(v^2 - \dot\rho^2) + 2(\boldsymbol{a}_0 \cdot \boldsymbol{x}_0 - v_0^2)\mathrm{e}^{t-t_0} + 2\int_{t_0}^{t} \mathrm{d}s\, \mathrm{e}^{t-s}\left(\frac{v^2}{2} + \frac{1}{\rho}\right). \tag{5}$$

This is seen as follows. Multiplying (2) by $\boldsymbol{x}$ and using the identity

$$\dot{\boldsymbol{a}} \cdot \boldsymbol{x} - \boldsymbol{a} \cdot \boldsymbol{x} = \mathrm{e}^t \frac{\mathrm{d}}{\mathrm{d}t}\left(\mathrm{e}^{-t}\left(\boldsymbol{a}\cdot\boldsymbol{x} - \frac{v^2}{2}\right)\right) - \frac{v^2}{2} \tag{6}$$

one obtains

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathrm{e}^{-t}\left(\boldsymbol{a}\cdot\boldsymbol{x} - \frac{v^2}{2}\right)\right) - \frac{v^2}{2} = \mathrm{e}^{-t}\left(\frac{v^2}{2} + \frac{1}{\rho}\right)$$

and this by integration gives

$$\boldsymbol{a}\cdot\boldsymbol{x} = \frac{v^2}{2} + \left(\boldsymbol{a}_0\cdot\boldsymbol{x}_0 - \frac{v_0^2}{2}\right)\mathrm{e}^{t-t_0} + \int_{t_0}^{t}\mathrm{d}s\,\mathrm{e}^{t-s}\left(\frac{v^2}{2} + \frac{1}{\rho}\right). \tag{7}$$

Equation (5) then follows by using $\rho\ddot\rho + \dot\rho^2 = \boldsymbol{a}\cdot\boldsymbol{x} + v^2$, which is obtained by twice differentiating the identity $\rho^2 = \boldsymbol{x}\cdot\boldsymbol{x}$ with respect to time.

From (5) we can show that $\ddot\rho\rho$ diverges. This follows by noting that $v^2 - \dot\rho^2$ is always non-negative, and on the other hand either ρ^{-1} is non-integrable or, by lemma 1, v^2 diverges.

Thus, there exists a positive constant k such that definitively one has

$$\ddot\rho > \frac{k}{\rho}. \tag{8}$$

But this gives a contradiction. Indeed the property $\ddot\rho > 0$ implies that $\dot\rho$ is monotonic increasing as $t \to t_c$, so that the limit of $\dot\rho$ exists; moreover, such a limit cannot be infinite because this would imply that $\dot\rho > 0$ (for t sufficiently close to t_c) and in turn this would imply ρ be increasing in contradiction with $\rho \to 0$. The existence of the limit implies that $\ddot\rho$ must be integrable; consequently, from inequality (8) it follows that ρ^{-1} is integrable too, and this gives a contradiction by (i) of lemma 1. This shows that it is impossible that t_c is finite.

We show now that even the case $t_c = +\infty$ gives a contradiction. In fact, suppose that for $t \to +\infty$ one has $\rho \to 0$. This implies that the solution is analytic for $t \in (\tilde t, +\infty)$ with a given $\tilde t$, and so from now on all times will be taken to be greater than $\tilde t$. On the other hand the following lemma, to be proven in section 3, holds.

Lemma 2. *If $\boldsymbol{x}$ is a solution of (2) such that $|\boldsymbol{x}| \to 0$ for $t \to +\infty$, then the following properties hold:*

(i) the function a^2 is integrable,

(ii) $\lim_{t\to+\infty} v^2 = +\infty$.

From lemma 2 the contradiction quickly follows. In fact one has

$$a^2 \geqslant a_r^2 \stackrel{\mathrm{def}}{=} (\ddot\rho + \dot\rho^2/\rho - v^2/\rho)^2$$

where a_r is the radial component of the acceleration $\boldsymbol{a}$. This, in particular, entails $\ddot\rho + \dot\rho^2/\rho \geqslant -a + v^2/\rho$, or also (by $\ddot\rho + \dot\rho^2/\rho = \rho^{-1}\frac{\mathrm{d}}{\mathrm{d}t}\rho\dot\rho$)

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho\dot\rho \geqslant v^2 - a\rho.$$

Now the integral $\int_{t_0}^{t}(v^2 - \rho a)\,\mathrm{d}s \to +\infty$ for $t \to +\infty$, because on the one hand one has (using the Cauchy estimate and (i) of lemma 2) $\int_{t_0}^{t} \rho a\,\mathrm{d}s < (\int_{t_0}^{t} a^2\,\mathrm{d}s)^{\frac{1}{2}}(\int_{t_0}^{t} \rho^2\,\mathrm{d}s)^{\frac{1}{2}} \leqslant K(t - t_0)^{\frac{1}{2}}$, with a suitable K, while, on the other hand, one has $\int_{t_0}^{t} v^2\,\mathrm{d}s \geqslant K'(t - t_0)$, with a suitable K', by (ii) of lemma 2. Thus, it follows that $\rho\dot{\rho} \to +\infty$, which implies that $\dot{\rho} \to +\infty$, in contradiction with $\rho \to 0$. This completes the proof. □

3. Proof of the lemmas

In the proof of the lemmas we make use of the relations

$$\boldsymbol{a}\cdot\boldsymbol{v} + \frac{1}{\rho} = \left(\boldsymbol{a}_0\cdot\boldsymbol{v}_0 + \frac{1}{\rho_0}\right)\mathrm{e}^{t-t_0} + \int_{t_0}^{t}\mathrm{d}s\,\mathrm{e}^{t-s}\left(a^2 - \frac{1}{\rho}\right) \tag{9}$$

$$a^2 = a_0^2\mathrm{e}^{2(t-t_0)} + \int_{t_0}^{t}\mathrm{d}s\,\mathrm{e}^{2(t-s)}\frac{2\boldsymbol{a}\cdot\boldsymbol{x}}{\rho^3} \tag{10}$$

which hold for solutions of (2). The proof is analogous to that of relation (5), which was obtained by multiplying (2) by $\boldsymbol{x}$ and using the identity (6); now (9) is obtained by multiplying (2) by $\boldsymbol{v}$ and using the identity $\dot{\boldsymbol{a}}\cdot\boldsymbol{v} = \frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{a}\cdot\boldsymbol{v}) - a^2$, while (10) is obtained by multiplying (2) by $\boldsymbol{a}$ and using the identity $\dot{\boldsymbol{a}}\cdot\boldsymbol{a} - a^2 = \mathrm{e}^{2t}\frac{\mathrm{d}}{\mathrm{d}t}(\mathrm{e}^{-2t}\frac{a^2}{2})$.

We will also make use of a generalized energy theorem which holds for equation (2), namely

$$E(t) = E(t_0) - \int_{t_0}^{t} a^2\,\mathrm{d}\,s \tag{11}$$

where

$$E \stackrel{\mathrm{def}}{=} \frac{v^2}{2} - \frac{1}{\rho} - \boldsymbol{a}\cdot\boldsymbol{v} \tag{12}$$

is the generalized energy E, which turns out to be a decreasing function of time. Relation (11) is obtained, as in the familiar case of Newton's equation, by multiplying (2) by $\boldsymbol{v}$ and using again the identity $\dot{\boldsymbol{a}}\cdot\boldsymbol{v} = \frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{a}\cdot\boldsymbol{v}) - a^2$. We now proceed to the proof of the lemmas.

Proof of lemma 1. The proof will be given by contradiction. As $1/\rho$ is integrable on $[t, t_\mathrm{c}]$, the Cauchy inequality gives

$$t_\mathrm{c} - t = \int_t^{t_\mathrm{c}}\mathrm{d}s\,\sqrt{\rho}\frac{1}{\sqrt{\rho}} \leqslant \left(\int_t^{t_\mathrm{c}}\mathrm{d}s\,\rho\right)^{\frac{1}{2}}\left(\int_t^{t_\mathrm{c}}\mathrm{d}s\,\frac{1}{\rho}\right)^{\frac{1}{2}}$$

or equivalently

$$\frac{t_\mathrm{c} - t}{(\int_t^{t_\mathrm{c}}\mathrm{d}s\,\rho)^{\frac{1}{2}}} \leqslant \left(\int_t^{t_\mathrm{c}}\mathrm{d}s\,\frac{1}{\rho}\right)^{\frac{1}{2}}.$$

The denominator at the lhs is estimated as follows. One has $\rho = -\int_t^{t_\mathrm{c}}\mathrm{d}s\,\dot{\rho}$ (because of $\rho(t_\mathrm{c}) = 0$), and so one finds

$$\int_t^{t_\mathrm{c}}\mathrm{d}s\,\rho = -\int_t^{t_\mathrm{c}}\mathrm{d}s\int_s^{t_\mathrm{c}}\mathrm{d}s'\,\dot{\rho} = \int_t^{t_\mathrm{c}}\mathrm{d}s\,(s - t)\dot{\rho} \leqslant \left(\frac{(t_\mathrm{c} - t)^{\frac{3}{2}}}{3^{\frac{1}{2}}}\right)\left(\int_t^{t_\mathrm{c}}\mathrm{d}s\dot{\rho}^2\right)^{\frac{1}{2}}$$

having again made use of the Cauchy estimate. Using this bound in the previous inequality and taking the fourth power of both members one finally gets

$$\frac{3(t_\mathrm{c} - t)}{\int_t^{t_\mathrm{c}}\mathrm{d}s\,\dot{\rho}^2} \leqslant \left(\int_t^{t_\mathrm{c}}\mathrm{d}s\,\frac{1}{\rho}\right)^2.$$

Denote now $k^2 \stackrel{\text{def}}{=} \limsup_{t\to t_c} \dot{\rho}^2$ and restrict t to an interval $[\bar{t}, t_c]$ such that $\dot{\rho}^2 < 3k^2$. Then one has

$$\frac{1}{k^2} \leqslant \left(\int_t^{t_c} \mathrm{d}s \frac{1}{\rho}\right)^2$$

i.e. $\limsup \dot{\rho}^2 > 1/(\int_t^{t_c} \mathrm{d}s\, 1/\rho)$; letting $t \to t_c$ this gives

$$\limsup_{t\to t_c} \dot{\rho}^2 = +\infty.$$

The first part of the lemma is thus proved.

We now proceed to the proof of the second part. The generalized energy relation (11), using $\boldsymbol{a}\cdot\boldsymbol{v} = \frac{\mathrm{d}}{\mathrm{d}t} v^2/2$, gives

$$-\mathrm{e}^t \frac{\mathrm{d}}{\mathrm{d}t}\left(\mathrm{e}^{-t}\frac{v^2}{2}\right) = E_0 + \frac{1}{\rho} - \int_{t_0}^t \mathrm{d}s\, a^2$$

from which, by integration from t_0 and t, one finds

$$\frac{v^2}{2} = \mathrm{e}^{t-t_0}\frac{v_0^2}{2} - E_0(\mathrm{e}^{t-t_0} - 1) + \int_{t_0}^t \mathrm{d}s\, \mathrm{e}^{t-s}\int_{t_0}^s \mathrm{d}s'\, a^2 - \int_{t_0}^t \mathrm{d}s\, \frac{\mathrm{e}^{t-s}}{\rho}. \tag{13}$$

From this, as $\int \mathrm{d}s\, \mathrm{e}^{t-s}/\rho \geqslant 0$, one obtains the following estimate:

$$\int_{t_0}^t \mathrm{d}s\, \mathrm{e}^{t-s}\int_{t_0}^s \mathrm{d}s'\, a^2 \geqslant \frac{v^2}{2} - \mathrm{e}^{t-t_0}\frac{v_0^2}{2} + E_0(\mathrm{e}^{t-t_0} - 1).$$

We now let $t \to t_c$ and remark that the limit of the lhs exists (we are dealing with an integral of a positive function) and that $v^2 \geqslant \dot{\rho}^2$; so one has

$$\lim_{t\to t_c}\int_{t_0}^t \mathrm{d}s\, \mathrm{e}^{t-s}\int_{t_0}^s \mathrm{d}s'\, a^2 \geqslant \limsup_{t\to t_c}\frac{v^2}{2} - \mathrm{e}^{t_c-t_0}\frac{v_0^2}{2} + E_0(\mathrm{e}^{t_c-t_0} - 1) = +\infty.$$

The other integral appearing at the rhs of (13) has instead a finite limit when t tends to t_c, because, by hypothesis, $1/\rho$ is integrable. Consequently, letting $t \to t_c$ in (13), one obtains that $\lim v^2$ exists and is infinite. □

Proof of lemma 2. We begin by showing that a^2 is integrable. Adding (7) and (9) one obtains

$$\boldsymbol{a}\cdot\boldsymbol{x} = (\boldsymbol{a}_0\cdot\boldsymbol{x}_0 - E_0)\mathrm{e}^{t-t_0} + E_0 + \int_{t_0}^t \mathrm{d}s\left[(\mathrm{e}^{t-s} - 1)a^2 + \mathrm{e}^{t-s}\frac{v^2}{2}\right] \tag{14}$$

which holds for all $t_0 < t < +\infty$. This implies that

$$\boldsymbol{a}\cdot\boldsymbol{x} \leqslant E = E_0 - \int_{t_0}^t \mathrm{d}s\, a^2 \tag{15}$$

for all t. In fact, if for some t_1 one had $\boldsymbol{a}\cdot\boldsymbol{x} > E$, taking initial data at time t_1 from (14) would get $\boldsymbol{a}\cdot\boldsymbol{x} > (\boldsymbol{a}_1\cdot\boldsymbol{x}_1 - E_1)\mathrm{e}^{t-t_1} + E_1$ (because the integral at the rhs of (14) is positive); in turn, in view of the identity $\ddot{\rho}\rho = \boldsymbol{a}\cdot\boldsymbol{x} + v^2 - \rho^2$, this implies that $\ddot{\rho}$ diverges exponentially fast to $+\infty$ in contradiction with $\rho \to 0$. Now using in (10) the estimate (15) one obtains

$$\frac{a^2}{2} \leqslant \mathrm{e}^{2(t-t_0)}\left(\frac{a_0^2}{2} + \int_{t_0}^t \mathrm{d}s\, \frac{\mathrm{e}^{2(t_0-s)}}{\rho^3}\left(E_0 - \int_{t_0}^s \mathrm{d}s'\, a^2\right)\right).$$

As a^2 has to be non-negative, for all $t_0 \in \mathbb{R}$ one has

$$\frac{a_0^2}{2} \geqslant \int_{t_0}^t \mathrm{d}s\, \frac{\mathrm{e}^{2(t_0-s)}}{\rho^3}\left(-E_0 + \int_{t_0}^s \mathrm{d}s'\, a^2\right) \qquad \forall t \geqslant t_0.$$

Now, the function $E_0 - \int_{t_0}^{s} \mathrm{d}s'\, a^2$ is a non-increasing function, so it has definitively a constant sign, so that the limit for $t \to +\infty$ of the integral at the rhs exists, and one gets

$$\frac{a_0^2}{2} \geqslant \int_{t_0}^{+\infty} \mathrm{d}s\, \frac{\mathrm{e}^{2(t_0-s)}}{\rho^3}\left(- E_0 + \int_{t_0}^{s} \mathrm{d}s'\, a^2\right).$$

Now one arrives at a contradiction supposing that a^2 is non-integrable. In fact, using lemma A of the appendix with $f = \frac{1}{\rho^3}\left(-E_0 + \int_{t_0}^{s} \mathrm{d}s' a^2\right)$, one finds $a_0^2 \to +\infty$ for $t_0 \to +\infty$. Furthermore, from (11), one has $E(t) \to -\infty$, so that there exists a t_0 for which $E_0 < 0$. Now, defining $\varepsilon_0 = \max_{t \leqslant t_0} \rho^3$, one has

$$\frac{a_0^2}{2} \geqslant \int_{t_0}^{+\infty} \mathrm{d}s\, \frac{\mathrm{e}^{2(t_0-s)}}{\rho^3}\left(- E_0 + \int_{t_0}^{s} \mathrm{d}s' a^2\right) \geqslant \frac{\int_{t_0}^{+\infty} \mathrm{d}s\, \mathrm{e}^{2(t_0-s)} \int_{t_0}^{s} \mathrm{d}s'\, a^2}{\varepsilon_0}$$

and changing the order of integration one gets

$$\frac{a_0^2}{2} \geqslant \frac{\int_{t_0}^{+\infty} \mathrm{d}s\, \mathrm{e}^{2(t_0-s)} a^2}{2\varepsilon_0}$$

which implies

$$\int_{t_0}^{+\infty} \mathrm{d}s\, \mathrm{e}^{2(t_0-s)}(a^2 - \varepsilon_0 a_0^2) \leqslant 0 \tag{16}$$

for all t_0. But now, defining $f(t) \stackrel{\mathrm{def}}{=} a^2(t)$ and $g(t) \stackrel{\mathrm{def}}{=} \max_{s \leqslant t} \rho^3$, one has that f and g are continuous functions and $\mathrm{e}^{-2t} f$ is clearly integrable. If, in addition, one has $f(t_0) = \boldsymbol{a}_0^2 \to +\infty$, so by lemma B of the appendix one gets $\int_{t_0}^{+\infty} \mathrm{d}s\, \mathrm{e}^{2(t_0-s)}(a^2 - \varepsilon_0 a_0^2) \geqslant 0$ in contradiction to (16).

We now turn to the proof of part (ii) of lemma 2: i.e. that $v^2 \to +\infty$. From definition (12) of the generalized energy one finds $\mathrm{e}^t \frac{\mathrm{d}}{\mathrm{d}t}(\mathrm{e}^t\, v^2/2) = -(E + 1/\rho)$ and by integration this gives

$$\frac{v^2}{2} = \mathrm{e}^{t-t_0}\left(\frac{v_0^2}{2} - \int_{t_0}^{t} \mathrm{d}s\, \mathrm{e}^{t_0-s}\left(E + \frac{1}{\rho}\right)\right).$$

As E is bounded below (recall that a^2 was just shown to be integrable) and ρ tends to zero, there exists $\bar{t}$ such that $E + 1/\rho > 0$ for $t > \bar{t}$. Taking $t_0 > \bar{t}$, from $v^2 \geqslant 0$ it follows that $v_0^2/2 \geqslant \int_{t_0}^{t} \mathrm{d}s\, \mathrm{e}^{t_0-s}(E + \frac{1}{\rho})$ for all $t > t_0$, which taking the limit $t \to +\infty$ gives

$$\frac{v_0^2}{2} \geqslant \int_{t_0}^{+\infty} \mathrm{d}s\, \mathrm{e}^{t_0-s}\left(E + \frac{1}{\rho}\right).$$

Thus, taking the limit $t_0 \to +\infty$ and using lemma A of the appendix one finds $v_0^2 \to +\infty$, i.e. part (ii) of the thesis. □

4. Comments

We have shown that it is impossible that the particle falls on the centre of force. It would be interesting to know whether there exist motions which remain bounded for all times. For motions on a line, Zin [5] showed not only that this is not the case, but also that, on the contrary, all motions are unbounded and have accelerations which asymptotically increase exponentially with time.

In the three-dimensional case it is known that there exist unbounded motions having vanishing asymptotic acceleration [6] (the so-called nonrunaway solutions), which represent, from the physical point of view, the scattering of an impinging particle by the centre of forces. If bounded solutions exist, they would correspond to the stable atom, but nothing is known presently.

From the physical point of view it would be interesting to consider the full relativistic Abraham–Lorentz–Dirac equations, because particles close to the centre of force can reach velocities close to the speed of light. The only rigorous result known to the present author is a theorem by Zin [5], which extends Eliezer's result to relativistic motions on a line.

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Appendix

Lemma A. Let $f(t) : [t_0, +\infty] \to \mathbb{R}$ be a continuous function for which $\lim_{t\to+\infty} f(t)$ exists. Then if $\mathrm{e}^{-t} f(t) \in L^1(t_0, +\infty)$ one has

$$\lim_{t\to+\infty} \int_t^{+\infty} \mathrm{d}s\, \mathrm{e}^{t-s} f(s) = \lim_{t\to+\infty} f(t).$$

Proof. The proof is a simple application of the L'Hospital theorem. In fact, one has

$$\int_t^{+\infty} \mathrm{d}s\, \mathrm{e}^{t-s} f(s) = \frac{\int_t^{+\infty} \mathrm{d}s\, \mathrm{e}^{-s} f(s)}{\mathrm{e}^{-t}}.$$

Now the integral $\int_t^{+\infty} \mathrm{d}s\, \mathrm{e}^{-s} f(s)$ tends to 0 as $t \to +\infty$, while, being $\mathrm{e}^{-s} f(s)$ continuous, the integral turns out to be a C^1 function of t. Thus, one can apply the L'Hospital theorem (see [7]), and then, taking the derivatives and passing to the limit, the thesis follows. □

Lemma B. Let $f(t), g(t) : [t_c, +\infty] \to \mathbb{R}$ be two continuous functions such that $f(t) \to +\infty$ and $g(t) \to 0$ for $t \to +\infty$. Suppose, in addition, that $\mathrm{e}^{-2t} f(t)$ is integrable. Then there exists $\bar{t}$ such that

$$\int_{\bar{t}}^{+\infty} \mathrm{d}s\, \mathrm{e}^{2(\bar{t}-s)}(g(\bar{t}) f(\bar{t}) - f(s)) < 0. \tag{B.1}$$

Proof. As $g(t)$ tends to zero and $f(t)$ diverges, there exists a time $\tilde{t}$ such that both $|g(t)| < 1/2$ and $f(t) > 1$ hold for $t > \tilde{t}$. So taking $t = t_0 > \tilde{t}$ in (B.1), as the integrand in (B.1) turns out to be negative for $s = t_0$, either there exists t_1 such that $g(t_0) f(t_0) > f(t_1)$, or the thesis is proven. Then taking in (B.1) $t = t_1$, by virtue of $g(t_1) f(t_1) < f(t_1)$, either there exists t_2 such that $g(t_1) f(t_1) > f(t_2)$ or the thesis is proven. In such a way we construct points $t_0, \ldots, t_n$ such that $g(t_k) f(t_k) > f(t_{k+1})$. So one has

$$f(t_n) < g(t_{n-1}) f(t_{n-1}) < g(t_{n-1}) g(t_{n-2}) f(t_{n-2}) < \cdots < f(t_0) \prod_{k=0}^{n-1} g(t_k).$$

This relation shows that the sequence $\{t_n\}$ cannot be continued indefinitely, as one has $f(t_n) > 1$ while $|g(t_0) \cdot g(t_1) \ldots g(t_n)| < 1/2^n$ tends to zero as $n \to +\infty$. So the sequence has to terminate for some t_n, and for $s > t_n$ one has $g(t_n) f(t_n) - f(s) < 0$; thus taking $\bar{t} = t_n$ one finds the thesis. □

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